MATHEMATICS OF QUANTUM MECHANICS - 057889

D. Fermi & M. Moscolari
Exercise classes: M. Cantoni

LECTURE PLAN

1. **17/09/2024, 10:15 - 12:15 (2 hours - D.F.).** Crisis of classical physics: black body radiation, photoelectric effect, Compton effect, atomic spectra, double slit experiment, wave-particle duality.
2. **20/09/2024, 08:15 - 10:15 (2 hours - D.F.).** Recap of classical physics: Newtonian, Lagrangian and Hamiltonian mechanics, theory of electromagnetism (Maxwell equations). Instability of classical atomic models.
3. **27/09/2024, 08:15 - 10:15 (2 hours - D.F.).** Optical/mechanical analogy, the Schrödinger equation, conservation of probability and Copenhagen interpretation. Average values and standard deviations of position and momentum operators, canonical commutation relations, Ehrenfest theorem, Heisenberg's uncertainty principle. Von Neumann postulates: states, observvables, dynamics, measurement.
4. **01/10/2024, 10:15 - 12:15 (2 hours - M.M.).** Definition and properties of inner product spaces, Pythagorean theorem and Cauchy-Schwarz inequality. Hilbert space: definition and examples. Orthonormal set and bases. Fourier coefficients. Isomorphism of Hilbert spaces and dimension. Projection theorem and property of dense subspaces.
5. **04/10/2024, 08:15 - 10:15 (2 hours - M.C. - Exercise class).** Analysis with Hamiltonian and Lagrangian formalism of a classical system composed by two point particle interacting with a central force. Short review on measure theory, convergence theorems and L^1 and L^2 spaces.
6. **08/10/2024, 10:15 - 12:15 (2 hours - M.M.).** Weak and strong convergence in Hilbert spaces. Bounded linear operators between normed spaces. Riesz lemma. Definition of Banach spaces and examples. Operators in Hilbert spaces. Uniform, strong and weak convergence of operators in Hilbert spaces. Bounded multiplication operators. Unbounded linear operators. Position operator. Graph and extension of linear operators. Definition of closed operators. Graph of closure of closable operators.
7. **11/10/2024, 08:15 - 10:15 (2 hours - M.C. - Exercise class).** Norm of bounded multiplication operators on $L^2(\mathrm{R})$. Observables for a single spin 1/2 quantum system.
8. **15/10/2024, 10:15 - 12:15 (2 hours - M.M.).** The position operator is closed on its maximal domain. Definition and examples of adjoint operator. Definition and example of symmetric and selfadjoint operators. The position operators is selfadjoint on its maximal domain. Definition of essential selfadjointness and example. Essentially selfadjoint operators has only one selfadjoint extension. Properties of adjoint operator. Definition and properties of kernel and range of linear operators. Basic criterion of selfadjointness.
9. **18/10/2024, 08:15 - 10:15 (2 hours - M.M.).** Definition of inverse operator. Resolvent set and resolvent operator. Digression on Banach valued analytic functions. Neumann series and properties of the resolvent operator. First characterization of the spectrum: point spectrum, continuous spectrum and residual spectrum. Analysis of the spectrum of selfadjoint operators.
10. **22/10/2024, 10:15 - 12:15 (2 hours - M.C. - Exercise class).** Properties of the adjoint operators. Example of adjoint and selfadjoint operators: momentum operator on the interval with different boundary conditions. Spectrum of the left and right shift operators.
11. **25/10/2024, 08:15 - 10:15 (2 hours - M.M.).** Weyl's criterion for the spectrum of a selfadjoint operator. Definition of unitary equivalent operators. Properties of unitary equivalent operators. The Fourier transform as an example of unitary transformation. Position operator and momentum operator are unitary equivalent through the Fourier transform. Definition and properties of orthogonal projections. Small digression on measure theory. Introduction to the spectral theorem. Holomorphic functional calculus for bounded operators.
12. **29/10/2024, 10:15 - 12:15 (2 hours - M.M.).** Continuous functional calculus and Borel functional calculus for bounded selfadjoint operators.
13. **05/11/2024, 10:15 - 12:15 (2 hours - M.M.).** Spectral theorem in the multiplication operator form for bounded and unbounded selfadjoint operators. Spectrum of multiplication operators and essential range. Definition of projection valued measure.
14. **08/11/2024, 8:15 - 10:15 (2 hours - M.C. - Exercise class).** Properties of projection valued measure. Computations of spectral measure of explicit operators.

There is no class on the 12th of November because of the midterm exam session.

15. **15/11/2024, 8:15 - 10:15 (2 hours - M.M.).** Spectral theorem in the projection valued measure form. Characterization of the spectrum using spectral measure: pure point spectrum, absolutely continuous spectrum, singular continuous spectrum. Characterization of the spectrum using spectral projections: discrete and essential spectrum. The problem of the existence of quantum dynamics. Strongly continuous one-parameter unitary groups. Selfadjoint operator as generator of strongly continuous one-parameter unitary groups.

16. **19/11/2024, 8:15 - 10:15 (2 hours - M.C. - Exercise class).** Example and computations of explicit spectral part of selfadjoint operators. Examples of strongly continuous one-parameter unitary groups.

17. **22/11/2024, 8:15 - 10:15 (2 hours - M.M.).** Characterization of the domain of selfadjoint operators using strongly continuous one-parameter unitary groups. Stone's theorem. Existence of quantum dynamics. Review of Von Neumann postulates: pure states as rays in Hilbert space; observables as selfadjoint operators; dynamics and brief mention to bound states and scattering states; measurements and spectral projections. Heisenberg's uncertainty relations for generic observables.

18. **26/11/2024, 10:15 - 12:15 (2 hours - D.F.).** The free particle model: dimensional and dimensionless formulation, self-adjoitness domain for the Hamiltonian operator, spectrum and projection valued measure. Explicit expressions for the integral kernels associated to the resolvent and to the unitary time evolution. Eigenfunction expansions, singular behavior along the diagonal, evaluation close the spectrum and connection with scattering. Dispersive estimates.

19. **29/11/2024, 08:15 - 10:15 (2 hours - D.F.).** The free particle model: wavepackets dynamics, evolution of average position and momentum, and of the related standard deviations; coherent superposition. The harmonic oscillator: classical and quantum Hamiltonians, dimensionless formulation.

20. **03/12/2024, 10:15 - 12:15 (2 hours - D.F.).** The harmonic oscillator: creation and annihilation operators, pure point spectrum, Rellich criterion, Ehrenfest theorem, localization, d-dimensional generalization and degeneracy.

21. **06/12/2024, 08:15 - 10:15 (2 hours - M.C. - Exercise class).** Computation of the integral kernel for the free Hamiltonian resolvent in dimensions 1 and 3, using the residue theorem. Computation of the integral kernel for the unitary group associated to the free Hamiltonian, using Gaussian integrals.

22. **10/12/2024, 10:15 - 12:15 (2 hours - D.F.).** The hydrogen atom: description of the model, classical mechanical formulation (Lagrangian, Hamiltonian, conserved quantities, qualitative description of motion). Quantum Hamiltonian: center of mass and reduced Hamiltonian, Kato-small perturbations and Kato-Rellich theorem, self-adjointness and boundedness from below. Compact operators.

23. **13/12/2024, 08:15 - 10:15 (2 hours - D.F.).** Lower bound for the spectrum of the hydrogen atom Hamiltonian via Hardy inequality. Weyl theorem: compact resolvent difference implies invariance of the essential spectrum. Essential spectrum of the hydrogen atom Hamiltonian. The virial theorem, existence of an infinite sequence of bound states accumulating at the continuous threshold. Ground state energy and Bohr state.

24. **17/12/2024, 10:15 - 12:15 (2 hours - D.F.).** The angular momentum operator: self-adjointness and spectral properties (via raising and lowering operators), spherical harmonics. Representation of the relative Coulomb Hamiltonian in polar coordinates and decomposition in angular momentum eigenspaces. Spectrum of the radial Hamiltonians. Point spectrum of the hydrogen atom Hamiltonian and degeneracy. Radial probability density for the ground state.

25. **20/12/2024, 08:15 - 10:15 (2 hours - M.C. - Exercise class).** The Hardy inequality, the Coulomb estimate, commutation rules for the angular momentum operator.

Last update: December 20, 2024